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Elliptical instability of a rapidly rotating, strongly stratified fluid

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The elliptical instability of a rotating stratified fluid is examined in the regime of a small Rossby number and order-one Burger number corresponding to rapid rotation and strong stratification. The Floquet problem describing the linear growth of disturbances to an unbounded, uniform-vorticity elliptical flow is solved using exponential asymptotics. The results demonstrate that the flow is unstable for arbitrarily strong rotation and stratification; in particular, both cyclonic and anticyclonic flows are unstable. The instability is weak, however, with growth rates that are exponentially small in the Rossby number. The analytic expression obtained for the growth rate elucidates its dependence on the Burger number and on the eccentricity of the elliptical flow. It explains, in particular, the weakness of the instability of cyclonic flows, with growth rates that are only a small fraction of those obtained for the corresponding anticyclonic flows. The asymptotic results are confirmed by numerical solutions of the Floquet problem. © *2009 American Institute of Physics*. [DOI: 10.1063/1.3177354]

I. INTRODUCTION

Elliptical instability, the three-dimensional instability of two-dimensional flows with elliptical streamlines, has been the focus of a great deal of research activity. The review in Ref. 1 discusses the main results up to 2002 and emphasizes the relevance of elliptical instability to a broad range of applications. One of these is the instability of two-dimensional vortices that are deformed elliptically by a large-scale strain flow. This is especially important for the dynamics of the atmosphere and ocean since this is characterized by an abundance of vortices that are deformed through either mutual interactions or the effect of large-scale flows. In this context, however, the planetary rotation and density stratification need to be taken into account.

Rotation and stratification clearly exert a strong influence on elliptical instability: Since this stems from the parametric resonance between the periodic fluctuations associated with the elliptical motion and the free waves supported by the flow, the dispersion relation of these waves is critical. In the presence of both rotation and stratification, the waves are inertia-gravity waves whose frequency is bounded from below by the minimum of the Coriolis parameter f and Brunt–Väisälä frequency N. As a consequence, a vortex of fixed vorticity ceases to be unstable by the subharmonic instability responsible for the simplest form of elliptical instability when both f and N exceed a certain threshold. As these parameters increase further, instabilities are limited to resonances of higher and higher order, leading to decreasing growth rates. This was clearly demonstrated by Miyazaki[2] on the basis of numerical solutions of the Floquet problem that models elliptic instability (see also Refs. 1 and 3). Further numerical results were obtained by McWilliams and Yavneh,[4] who concentrated on the regime of rapid rotation and strong stratification with $N>f$ most relevant to the atmosphere and ocean. Their broad motivation was the role that instabilities play in the generation of inertia-gravity-wave-like motion, and the resulting breakdown of the nearly geostrophic and hydrostatic balance that is typical of much of the atmosphere and ocean. The present paper shares the same motivation. It re-examines the elliptical instability of a rotating stratified fluid and derives explicit analytical results in the limit of fast rotation $f \gg \Omega$ and strong stratification $N \gg \Omega$, where Ω denotes the (relative) vorticity of the flow.

Several recent papers[5–8] have demonstrated in specific examples that instabilities of well-balanced basic flows to inertia-gravity-wave perturbations (or perturbations related to similar fast waves) have growth rates that are exponentially small in the Rossby number, here proportional to $\Omega/f \ll 1$.[9] Theoretical arguments[10,11] indicate that this is a generic property, and the elliptical instability examined in this paper is no exception. In this case, the exponential smallness can be roughly understood by noting that in the manner typical of parametric instabilities,[12] the growth rates of the elliptical instability can be expected to be proportional to Ω^n, where n is the order of the resonance. Since, as pointed out by Miyazaki,[2] the minimum n is of the order f/Ω (for $N>f$), this leads to the conclusion that growth rates are beyond all orders in the Rossby number. To go further than this rough argument and provide an estimate for the growth rate requires the exponential-asymptotics analysis of the Floquet problem relevant to the elliptical instability. We carry out this analysis and, rather than relying on general asymptotic results for Hill's equations,[13] directly relate the growth of solutions to the occurrence of a Stokes phenomenon[14] which we capture using a combination of WKB expansion and matched asymptotics in complex time. The analytical results are confirmed by the numerical solutions of the Floquet problem.

One of the issues which our treatment clarifies is the difference between the instability of cyclonic and anticyclonic vortices in the presence of both rapid rotation and strong stratification. Cyclones have been recognized as less

unstable than anticyclones, to the extent that McWilliams and Yavneh[4] considered only the instability of the latter. We show that the cyclones are in fact linearly unstable, with growth rates that have the same exponential dependence on the Rossby number as the corresponding anticyclones but differ by a factor which, although formally of order one, turns out to be numerically very small.

The plan of this paper is as follows. In Sec. II, we formulate the problem of elliptical instability in a rotating stratified fluid modeled using the Boussinesq approximation. We use the simplest instance of elliptical instability, that of an unbounded elliptical vortex with uniform vorticity. This makes it possible to seek global solutions in the form of plane waves with time-periodic wavevector and an amplitude that satisfies Hill's equation. (Results for this particular case have a much broader appeal, however, since an identical Hill's equation arises when the stability of more general elliptical vortices is examined locally using the geometric-optics technique.[15–17]) The Floquet problem associated with Hill's equation is solved asymptotically in Sec. III in the limit of fast rotation and strong stratification, with the eccentricity of the elliptical streamlines assumed of order one. For simplicity, we also make the hydrostatic approximation assuming that $N \gg f$ and an order-one Burger number. The asymptotic derivation is only sketched in Sec. III, with technical details relegated to Appendix A. The asymptotic results are confirmed by direct numerical solution of the Floquet problem in Sec. IV. The effect of a finite N/f is also briefly examined there.

II. FORMULATION

We consider the stability of a horizontal elliptical flow in a three-dimensional stratified fluid, with constant Brunt–Väisälä frequency N, rotating about the vertical axis at rate $f/2>0$. The flow's streamfunction, velocity, and vorticity are written as

$$\Psi = -\tfrac{1}{2}(bx^2 + ay^2), \quad \boldsymbol{U} = (ay, -bx, 0), \quad \text{and} \quad \Omega = -(a+b), \tag{2.1}$$

where $ab>0$. We define

$$\varsigma = \operatorname{sgn} a = \operatorname{sgn} b$$

and note that the flow is anticyclonic for $\varsigma=1$, and cyclonic for $\varsigma=-1$. Three dimensionless parameters characterize the flow, namely,

$$e = \sqrt{a/b}, \quad \epsilon = \sqrt{ab}/f, \quad \text{and} \quad f/N, \tag{2.2}$$

which are recognized as the aspect ratio of the elliptical flow, a Rossby number, and the Prandtl ratio. We assume that $e>1$ without loss of generality.

Perturbations to the flow (2.1) take the form of plane waves with time-dependent wavevector, with each field written as

$$u(\boldsymbol{x},t) = \hat{u}(t) e^{i\boldsymbol{k}(t)\cdot\boldsymbol{x}}, \tag{2.3}$$

where the wavevector $\boldsymbol{k}=(k,l,m)$ satisfies

$$\dot{k} = bl, \quad \dot{l} = -ak, \quad \text{and} \quad \dot{m} = 0, \tag{2.4}$$

the overdot denoting differentiation with respect to t. In what follows, we use a dimensionless time variable obtained by taking $(ab)^{-1/2}$ as a reference time. In terms of this variable, the solutions to Eq. (2.4) have the simple form

$$k = k_0 \cos t, \quad l = -\varsigma e k_0 \sin t, \quad \text{and} \quad m = m_0, \tag{2.5}$$

where k_0 and m_0 are constant. The stability of Eq. (2.1) depends on the behavior of the amplitudes $\hat{u}(t)$, $\hat{v}(t)$, etc., as $t\to\infty$. These satisfy a set of ordinary differential equations with time-periodic coefficients. Following McWilliams and Yavneh,[4] this set can be conveniently reduced to a single second-order equation for the amplitude of the vertical component of the vorticity $\hat{\zeta}=i(l\hat{v}-k\hat{u})$. See Appendix A for details. Assuming that the perturbation potential vorticity vanishes, this equation reduces to

$$\ddot{\zeta} + \frac{2\varsigma klm^2(e-e^{-1})}{\kappa^2(k^2+l^2)}\dot{\zeta} + \frac{1}{\epsilon^2}\left\{[1-\varsigma\epsilon(e+e^{-1})\left(1-\frac{2\varsigma\epsilon e k_0^2}{k^2+l^2}\right)\frac{m^2}{\kappa^2} + \frac{N^2(k^2+l^2)}{f^2\kappa^2}\right\}\zeta = 0, \tag{2.6}$$

where $\kappa^2=k^2+l^2+m^2$ and we have omitted the hat on the amplitude ζ. Four dimensionless parameters appear in this equation: the three flow parameters (2.2), and the initial aspect ratio m_0/k_0 of the perturbation.

Most of this paper focuses on a limiting case of Eq. (2.6) obtained by making the hydrostatic approximation. This assumes that $m_0 \gg k_0$ and $N \gg f$ while

$$\mu = \frac{fm_0}{Nk_0} = O(1). \tag{2.7}$$

This is the regime most relevant to the dynamics of the atmosphere and oceans since the condition $N\gg f$ is verified while, as we demonstrate below, the largest growth rates of the elliptical instability correspond to $\mu=O(1)$. The parameter μ can be recognized as the inverse square root of a Burger number; it can be interpreted as the aspect ratio of the perturbation scaled by f/N as is natural in rapidly rotating, strongly stratified fluids.

In the hydrostatic approximation, κ^2 is approximated by m^2, and Eq. (2.6) reduces to

$$\ddot{\zeta} + \frac{2\varsigma kl(e-e^{-1})}{k^2+l^2}\dot{\zeta} + \frac{1}{\epsilon^2}\left\{[1-\varsigma\epsilon(e+e^{-1})\left(1-\frac{2\varsigma\epsilon e k_0^2}{k^2+l^2}\right) + \frac{N^2(k^2+l^2)}{f^2m^2}\right\}\zeta = 0. \tag{2.8}$$

Using Eq. (2.5) and defining $\psi>0$ by

$$e^2 = 1+\psi^2, \tag{2.9}$$

we rewrite this equation as

$$\ddot{\zeta} - p(t)\dot{\zeta} + \frac{1}{\epsilon^2}[\omega^2(t) - \epsilon q(t) + \epsilon^2 r(t)]\zeta = 0. \tag{2.10}$$

Here

$$\omega^2 = 1 + \frac{N^2(k^2+l^2)}{f^2 m^2} = 1 + \mu^{-2}(1 + \psi^2 \sin^2 t) \tag{2.11}$$

can be recognized as the square of the inertia-gravity-wave frequency (nondimensionalized by f). We have also introduced

$$p(t) = \frac{\psi^2 \sin(2t)}{1 + \psi^2 \sin^2 t}, \quad q(t) = \mathrm{s}\left(e + e^{-1} + \frac{2e}{1 + \psi^2 \sin^2 t}\right), \tag{2.12}$$

$$\text{and} \quad r(t) = \frac{2(e^2+1)}{1 + \psi^2 \sin^2 t}.$$

Equation (2.10) is a Hill equation, with coefficients that are π-periodic in t. Its stability is determined using the Floquet theory for Hill equations.[12] We briefly recall the main result of this theory. Let $\zeta_1(t)$ and $\zeta_2(t)$ denote two linearly independent solutions of Eq. (2.10), and let us form the column vector $\boldsymbol{\zeta}(t)=[\zeta_1(t),\zeta_2(t)]^{\mathrm{T}}$. The π-periodicity of the coefficients in Eq. (2.10) implies that

$$\boldsymbol{\zeta}(t+\pi) = M\boldsymbol{\zeta}(t)$$

for some constant matrix M. The eigenvalues λ of M are then the Floquet multipliers, and two fundamental solutions can be found for which

$$\zeta(t) = e^{\sigma t}\phi(t), \tag{2.13}$$

where

$$\sigma = \frac{1}{\pi}\log \lambda \tag{2.14}$$

is the Floquet exponent, and $\phi(t)$ is π-periodic. Note that the form of the coefficient of $\dot{\zeta}$ ensures that the two multipliers satisfy $\lambda_1\lambda_2=1$. Instability occurs when one of these is such that $|\lambda|>1$ or, equivalently, Re $\sigma>0$. The matrix M is computed by relating $\boldsymbol{\zeta}$ and $\dot{\boldsymbol{\zeta}}$ at two times t and $t+\pi$. Here we choose $t=-\pi/2$ and compute M as

$$M = [\boldsymbol{\zeta}(\pi/2), \dot{\boldsymbol{\zeta}}(\pi/2)][\boldsymbol{\zeta}(-\pi/2), \dot{\boldsymbol{\zeta}}(-\pi/2)]^{-1}. \tag{2.15}$$

Our aim is to determine the largest values of the growth rate Re σ for Eq. (2.10) analytically in the fast-rotation regime $\epsilon \ll 1$, with $N \gg f$, $\mu=O(1)$, and $\psi=O(1)$. In this regime Eq. (2.10) resembles the Hill equations with large parameters whose stability was studied by Weinstein and Keller[13] using a mapping to parabolic cylinder functions. However, there are difficulties in applying their results directly, related to the presence of a first derivative term that is singular for the complex values of t such that $k^2+l^2=0$. We have therefore found it simpler to develop a different approach, combining WKB analysis with complex-time matching. This approach, which has the advantage of demonstrating the link between the instability and the Stokes phenomenon,[14,18] is described in Sec. III and in Appendix B. The analytic results obtained there are confirmed and extended to finite N/f in Sec. IV by solving Eq. (2.10) numerically.

III. WKB ANALYSIS

A WKB solution of the form

$$\zeta(t) = A(t)e^{i\theta(t)/\epsilon} \tag{3.1}$$

can be introduced into Eq. (2.10) and the (real) functions $A(t)$ and $\theta(t)$ can be sought as the asymptotic series

$$A = A_0 + \epsilon A_1 + \cdots \quad \text{and} \quad \theta = \theta_0 + \epsilon\theta_1 + \cdots. \tag{3.2}$$

Introducing Eqs. (3.1) and (3.2) into Eq. (2.10), we find at leading order that

$$\theta_0(t) = \int_{-\pi/2}^{t} \omega(t')dt'. \tag{3.3}$$

At the next order, we have

$$\frac{\dot{A}_0}{A_0} = -\frac{\dot{\omega}}{2\omega} + \frac{p}{2}, \tag{3.4}$$

$$\dot{\theta}_1 = -\frac{q}{2\omega}. \tag{3.5}$$

We note that $A_0(t)$ is π-periodic since $p(t)$ has zero mean in $[-\pi/2,\pi/2]$. Pursuing the computation to higher orders suggests that all the $A_n(t), n=0,1,\ldots$ are π-periodic. This can be confirmed by considering the Wronskian of the pair of solutions

$$\zeta = A(t)e^{i\theta(t)/\epsilon} \quad \text{and} \quad \bar{\zeta} = A(t)e^{-i\theta(t)/\epsilon}. \tag{3.6}$$

This function, defined by $W=\dot{\zeta}\bar{\zeta}-\zeta\dot{\bar{\zeta}}=2i\epsilon^{-1}A^2\dot{\theta}$, satisfies $\dot{W}=pW$ and hence is π-periodic. Now, the recurrence relations for A_n and θ_n [of which Eqs. (3.4) and (3.5) are the first terms] give $\dot{\theta}_n$ in terms of periodic functions and the A_m for $m<n$. The periodicity of the A_n then follows inductively from that of the Wronskian.

The periodicity of the asymptotic series for $A(t)$ implies that there is no instability to any algebraic order in ϵ. To all algebraic orders, the fundamental solutions are given by Eq. (3.6) and the Floquet multipliers are simply $\lambda=\exp[\pm i\theta(\pi/2)]$ [taking $\theta(-\pi/2)=0$] and satisfy $|\lambda|=1$. Any instability is necessarily due to exponentially small effects. To capture the instability and estimate its growth rate, we therefore need to go beyond the standard WKB analysis and examine how exponentially small terms alter the form (3.6) of the solutions. These terms arise through a Stokes phenomenon associated with the presence of complex turning points, that is, complex values of t for which $\omega(t)=0$. From these points emanate lines of the complex t-plane, termed Stokes lines, across which solutions defined by asymptotic series such as Eq. (3.2) acquire an additional exponentially small multiple of other linearly independent solutions. (See Refs. 19–22 for the analysis of the Stokes phenomenon in a variety of problems.)

Since the matrix M in Eq. (2.15) depends only on the solution vector $\boldsymbol{\zeta}$ at $t=\pm\pi/2$, the only Stokes lines relevant to our problem are those crossing the real t-axis between $-\pi/2$ and $\pi/2$. As explained in Appendix B, there is only one such line, namely (a segment of) the line Re $t=0$. Across this line, the asymptotic solutions ζ and $\bar{\zeta}$ in Eq. (3.6) switch on an exponentially small multiple of $\bar{\zeta}$ and ζ, respectively. More specifically, Eq. (3.6) defines a pair of independent solutions that is valid for $-\pi/2\leq t<-\delta$, for some $\epsilon^{1/2}\ll\delta\ll 1$. When crossing Re $t=0$, these solutions gain additional terms; for $\delta<t\leq\pi/2$, they become

$$\zeta = A(t)[e^{i\theta(t)/\epsilon} + Se^{-i\theta(t)/\epsilon}] \quad \text{and} \quad \bar{\zeta} = A(t)[e^{-i\theta(t)/\epsilon} + \bar{S}e^{i\theta(t)/\epsilon}], \tag{3.7}$$

where S is an exponentially small constant. This constant, the Stokes multiplier, is computed in Appendix B and its modulus is given explicitly in Eq. (3.12) below. Note that $A(t)$ and $\theta(t)$ appearing in Eqs. (3.6) and (3.7) are defined by the asymptotic series (3.2), which diverge and must be truncated. The theory of the Stokes phenomenon[19] shows that optimal truncation, in which the series are truncated near their smallest term, leads to truncation errors that are asymptotically smaller than the Stokes multiplier S. Thus Eqs. (3.6) and (3.7) can be interpreted consistently by regarding $A(t)$ and $\theta(t)$ as defined by their asymptotic series truncated optimally. We adopt this interpretation and carry out our analysis neglecting all $o(S)$ terms.

The Stokes multiplier S turns out to determine completely the leading-order form of the maximum growth rate of the instability. To show this, we compute the matrix M in Eq. (2.15) taking $\boldsymbol{\zeta}=(\zeta,\bar{\zeta})^T$. We first have that

$$[\boldsymbol{\zeta}(\pi/2),\dot{\boldsymbol{\zeta}}(\pi/2)] = \begin{pmatrix} A(e^{i\theta/\epsilon}+Se^{-i\theta/\epsilon}) & (i\epsilon^{-1}\dot{\theta}A+\dot{A})e^{i\theta/\epsilon}+S(-i\epsilon^{-1}\dot{\theta}A+\dot{A})e^{-i\theta/\epsilon} \\ A(e^{-i\theta/\epsilon}+\bar{S}e^{i\theta/\epsilon}) & (-i\epsilon^{-1}\dot{\theta}A+\dot{A})e^{-i\theta/\epsilon}+\bar{S}(i\epsilon^{-1}\dot{\theta}A+\dot{A})e^{i\theta/\epsilon} \end{pmatrix},$$

where A, $\dot{A}$, θ, and $\dot{\theta}$ are evaluated at $t=\pi/2$. Similarly,

$$[\boldsymbol{\zeta}(-\pi/2),\dot{\boldsymbol{\zeta}}(-\pi/2)] = \begin{pmatrix} A & i\epsilon^{-1}\dot{\theta}A+\dot{A} \\ A & -i\epsilon^{-1}\dot{\theta}A+\dot{A} \end{pmatrix}.$$

Here A, $\dot{A}$, and $\dot{\theta}$ can be evaluated at $t=\pi/2$, as above, since these functions (defined by their optimally truncated series expansion) are π-periodic. Computing Eq. (2.15) gives the simple result

$$M = \begin{pmatrix} e^{i\theta/\epsilon} & Se^{-i\theta/\epsilon} \\ \bar{S}e^{i\theta/\epsilon} & e^{-i\theta/\epsilon} \end{pmatrix}. \tag{3.8}$$

Here $\theta=\theta(\pi/2)$ or, more generally, $\theta=\theta(\pi/2)-\theta(-\pi/2)$ which accommodates any convention for the arbitrary choice of $\theta(-\pi/2)$.

It is now readily checked that up to $o(S)$ errors, the Floquet multipliers, that is, the eigenvalues λ of Eq. (3.8), satisfy $|\lambda|=1$ unless M is degenerate or nearly degenerate, in the sense that $\exp(i\theta)=\exp(-i\theta)+O(S)$. Thus the condition for instability $|\lambda|>1$ requires that $\exp(i\theta)$ is exponentially close to ± 1.[23] We therefore suppose that the parameters e, μ, and ϵ are such that

$$e^{i\theta/\epsilon} = \pm(1+iT)+o(S) \tag{3.9}$$

for some $T\in\mathbb{R}$ of a similar order of magnitude as S. The Floquet multipliers obtained from Eq. (3.8) are then given by

$$\lambda = \pm(1+\sqrt{|S|^2-T^2})+o(S) \quad \text{and} \quad \lambda = \pm(1-\sqrt{|S|^2-T^2})+o(S). \tag{3.10}$$

Clearly, one of these multipliers satisfies $|\lambda|>1$, and the flow is unstable, provided that $-|S|\leq T\leq|S|$, that is, in exponentially narrow instability bands. The corresponding growth rate $\sigma=\sqrt{|S|^2-T^2}/\pi+o(S)$ is maximum at the center of these bands, for $T=0$, and is given by

$$\sigma_{\max}\sim\frac{|S|}{\pi}. \tag{3.11}$$

The computation of S is carried out in Appendix B. There we show that

$$|S| = e^{-\alpha/\epsilon+\mathrm{s}\beta}, \tag{3.12}$$

where

$$\alpha = \frac{2}{\mu}\int_0^{\sqrt{1+\mu^2}/\psi}\sqrt{\frac{1+\mu^2-\psi^2x^2}{1+x^2}}dx \tag{3.13}$$

and

$$\beta = \mu\,⨍_0^{\sqrt{1+\mu^2}/\psi}\left(e+e^{-1}+\frac{2e}{1-\psi^2x^2}\right)\times\frac{dx}{\sqrt{(1+\mu^2-\psi^2x^2)(1+x^2)}}. \tag{3.14}$$

Here ⨍ denotes the Cauchy principal value of the integral, whose integrand is singular at $x=1/\psi$.

Figure 1 shows the values of α and β as functions of e and μ. Some conclusions can be drawn from the figure and

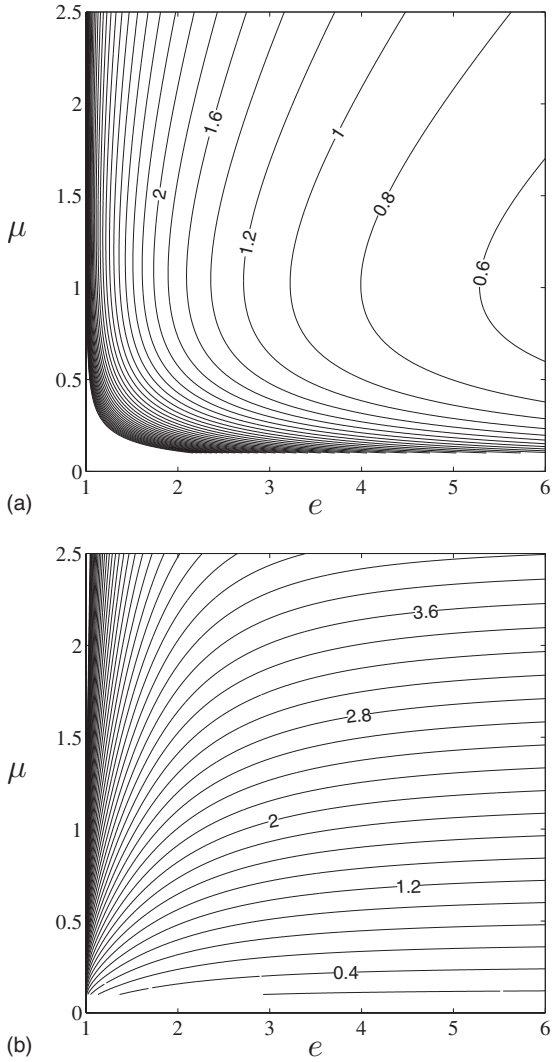

FIG. 1. Parameters α and β governing the maximum growth rates according to Eqs. (3.11) and (3.12) as functions of e and μ.

the examination of the explicit expressions (3.13) and (3.14). First, $\alpha \to \infty$ in the limits of both small and large μ; specifically $\alpha = O(\mu^{-1})$ as $\mu \to 0$ and $\alpha = O(\log \mu)$ as $\mu \to \infty$. This suggests, as is confirmed by Fig. 1, that the largest growth rates are attained for $\mu = O(1)$. Thus, the aspect ratio of the perturbations that grow as a result of the elliptical instability of vortices should be expected to be the Prandlt ratio: $m_0/k_0 = O(N/f)$. Second, the behavior of α for small and large eccentricity is given by

$$\alpha \sim -\frac{2\sqrt{1+\mu^2}}{\mu}\left[\log \psi + 1 - 2\log 2 - \frac{1}{2}\log(1+\mu^2)\right] \quad \text{as} \quad \psi \to 0, \tag{3.15}$$

$$\alpha \sim \frac{(1+\mu^2)\pi}{2\mu\psi} \quad \text{as} \quad \psi \to \infty. \tag{3.16}$$

The large-ψ expression (3.16) can actually be used to estimate α for values of ψ as small as 1, which makes it very useful. [For $\mu = 1$, for instance, the errors in Eq. (3.16) are 15%, 10%, and 5% for $\psi = 1$, 1.5, and 2, respectively.] This expression shows, in particular, that the largest growth rates are attained precisely for $\mu \sim 1$ when ψ is large. Third, the obvious fact that $\beta > 0$ shows that anticyclonic flows ($\varsigma = 1$) are more unstable than cyclonic flows ($\varsigma = -1$). According to Eq. (3.12), the growth rate in an anticyclonic flow is a factor $\exp(2\beta)$ larger than the growth rate of the corresponding cyclonic flow. Formally, this is an $O(1)$ factor, but the typical values of β are such that it is numerically very small so that the instability of cyclones is exceedingly weak and probably negligible in most circumstances. Note that because β is a decreasing function of e, the asymmetry between cyclones and anticylones is the largest for small eccentricity.

Formulas (3.11)–(3.14) give completely explicitly expressions for the maximum growth rates of the elliptical instability in terms of the three parameters ϵ, μ and e (recall that $\psi = \sqrt{e^2-1}$). These growth rates are achieved when the three parameters are related in such a way that $\exp(i\theta/\epsilon) = \pm 1$, that is,

$$\theta = n\pi\epsilon, \quad n = 1,2,\ldots. \tag{3.17}$$

This condition can be recognized as a resonance condition between the phase of the inertia-gravity oscillations and the period of rotation around the elliptical vortex (2π in the dimensionless time used here).

The growth rates can be written more directly in terms of ϵ, μ, and e by solving Eq. (3.17) perturbatively, with $\theta = \theta_0 + \epsilon\theta_1 + \cdots$, and θ_0 and θ_1 obtained from Eqs. (3.3) and (3.5). This gives the approximate position of the instability bands as well as their width. To leading order, the instability bands are centered at values of e and μ satisfying

$$\theta_0 = \frac{1}{\mu}\int_{-\pi/2}^{\pi/2} \sqrt{1+\mu^2+\psi^2\sin^2 t}\, dt = \frac{2}{\mu}\int_0^1 \sqrt{\frac{1+\mu^2+\psi^2x^2}{1-x^2}}\, dx = n\pi\epsilon, \quad n = 1,2,\ldots. \tag{3.18}$$

The computation of the correction $\epsilon\theta_1$ is more involved. Note that it is, in principle, needed to obtain a leading-order approximation to the growth rate Re σ as a function of e and μ. This is because the error on α needs to be $o(\epsilon)$, which requires to approximate the resonance values of e and μ with an $o(\epsilon)$ error also. We do not pursue these detailed computations here: since the values of e and μ satisfying the resonance condition (3.17) are ϵ-close together, Eq. (3.11) provides a useful approximation to the growth rates of the instability without the need to locate the resonances accurately. This is demonstrated in Sec. IV where we compare the prediction (3.11) with numerical solutions of the Floquet problem associated with Eq. (2.8).

Note that the instability-band width can be deduced directly from expression (3.18) for θ_0. For fixed ϵ and e, for instance, T in Eq. (3.9) can be written as $T = \epsilon^{-1}\Delta\mu\,\partial_\mu\theta$, where $\Delta\mu$ is the distance between μ and the resonant values, and the derivative is evaluated at resonance. According to Eq. (3.10), the instability bandwidth is therefore $\Delta\mu = 2\epsilon|S|/\partial_\mu\theta$, where θ can be approximated by θ_0.

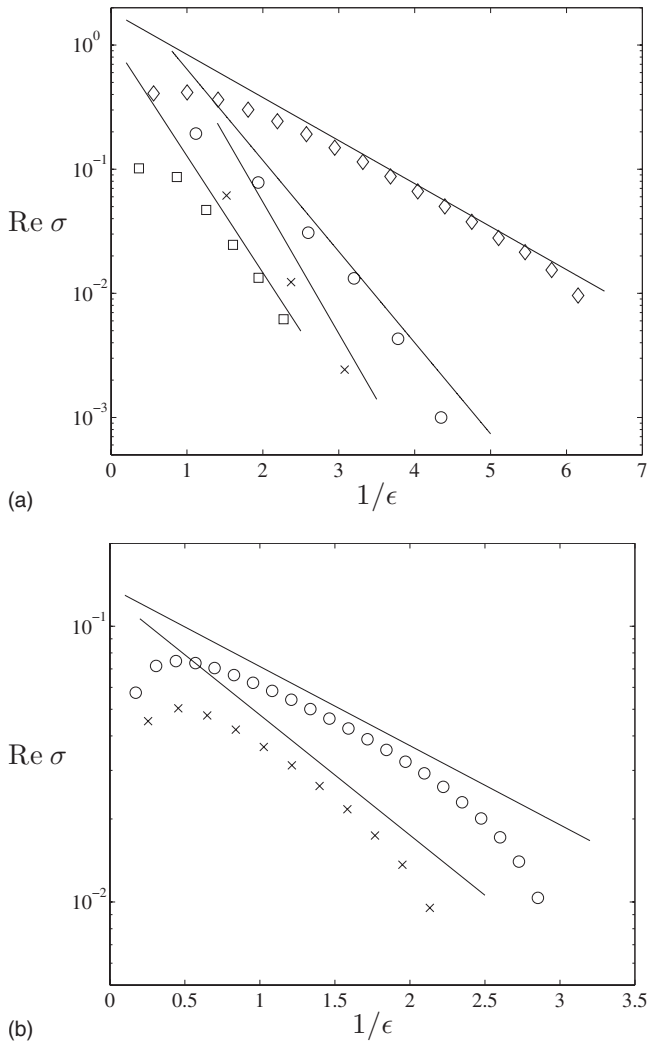

FIG. 2. Numerical estimates of the local maxima of the growth rates Re σ as functions of the inverse Rossby number $1/\epsilon$. (a) Anticyclonic flows with $e=1.5$, $\mu=1$ (×); $e=2$, $\mu=1$ (○); $e=2$, $\mu=0.5$ (□); and $e=4$, $\mu=1$ (◇). (b) Cyclonic flows with $e=4$, $\mu=0.5$ (×); and $e=6$, $\mu=0.5$ (○). The straight lines in these linear-logarithmic plots indicate the exponential dependence predicted by asymptotic estimate (3.11).

IV. COMPARISON WITH NUMERICAL RESULTS

We have solved the Floquet problem associated with Eq. (2.10) for the amplitude ζ numerically using MATLAB's standard Runge–Kutta solver. The growth rates Re σ obtained in this manner are compared to the asymptotic estimate (3.11) for $\sigma_{\max}$. To emphasize the exponential dependence on the inverse Rossby number $1/\epsilon$, it is convenient to display Re σ as a function of $1/\epsilon$ for fixed values of μ and of e. Figures 2(a) and 2(b) summarize the results obtained for several anticyclonic vortices (with $\varsigma=1$) and cyclonic vortices ($\varsigma=-1$), respectively. They display in linear-logarithmic coordinates the local maxima of the growth rates obtained numerically as ϵ is varied. The corresponding asymptotic predictions (3.11) are shown as line segments. The maximum growth rates displayed correspond to resonant values of $1/\epsilon$ for fixed e and μ. Figures 3 and 4 illustrate the complete structure of the instability bands that surround the resonant values by showing all the nonzero growth rates obtained numerically, in linear coordinates and for the parameters of Figs. 2(a) and 2(b), respectively.

The figures confirm the validity of our asymptotic estimate. They also suggest that this estimate remains useful for moderately small values of ϵ, say $\epsilon \lesssim 1/2$. Note that the dimensional growth rates are obtained by multiplying σ by $\sqrt{ab}$ which is related to the relative vorticity $|\Omega|=a+b$ of the flow by $\sqrt{ab}=|\Omega|/(e+e^{-1})$. As expected from our asymptotics, the growth rates in the case of cyclonic flows are exceedingly small for $\epsilon \ll 1$ even for the large eccentricities used in Fig. 4. Nonetheless, our results clarify the fact that all elliptical flows are unstable, regardless of the sense of the rotation, of its strength, and of the strength of the stratification. Note that the match between asymptotic and numerical results appears to degrade for small ϵ (i.e., large $1/\epsilon$), especially for cyclonic flows; this is because the smallness of both the growth rate and instability bandwidth makes the maximum growth rate delicate to estimate numerically. In particular, the agreement between asymptotic and numerical result could be improved by using a finer resolution in $1/\epsilon$.

The separation between instability bands can be estimated from the asymptotic formula (3.18): in terms of the varying $1/\epsilon$ used in the figures, it is given by

$$\gamma = \frac{\pi\mu}{2\int_0^1 \sqrt{\dfrac{1+\mu^2+\psi^2x^2}{1-x^2}}\,dx}.$$

Evaluating this quantity for the parameters chosen for the figures gives $\gamma=0.62$, 0.54, 0.31, and 0.34 for the parameters of Figs. 3(a)–3(d), and $\gamma=0.18$ and 0.12 for the parameters of Figs. 4(a) and 4(b), in good agreement with the numerical results.

Our derivation of an asymptotic expression for the growth rate makes the hydrostatic approximation, which assumes that $N \gg f$, $m_0 \gg k_0$, and $\mu = fm_0/(Nk_0) = O(1)$. This assumption, which could be relaxed, is made because it corresponds to the regime most relevant to atmospheric and oceanic flows; it is consistent in the sense that the growth rates obtained are maximized for $\mu = O(1)$ and decay rapidly for $\mu \gg 1$ or $\mu \ll 1$. To test the sensitivity of the results to the hydrostatic approximation, we have solved the Floquet problem numerically for the full, unapproximated Eq. (2.6) for moderately large values of N/f and m_0/k_0. The results obtained for $\mu=1$ and $e=4$ are displayed in Fig. 5. This compares the growth rate obtained in the hydrostatic approximation with those obtained for $N/f=m_0/k_0=3$ and 6. Except for $\epsilon \gtrsim 1$, there is relatively little difference between the results: The maximum growth rates fall on the same curve, well described by the hydrostatic asymptotics. Of course, the location of the instability bands changes depending on S, but this is not significant since they would also change if m_0/k_0 was varied independently of N/f, as is physically relevant.

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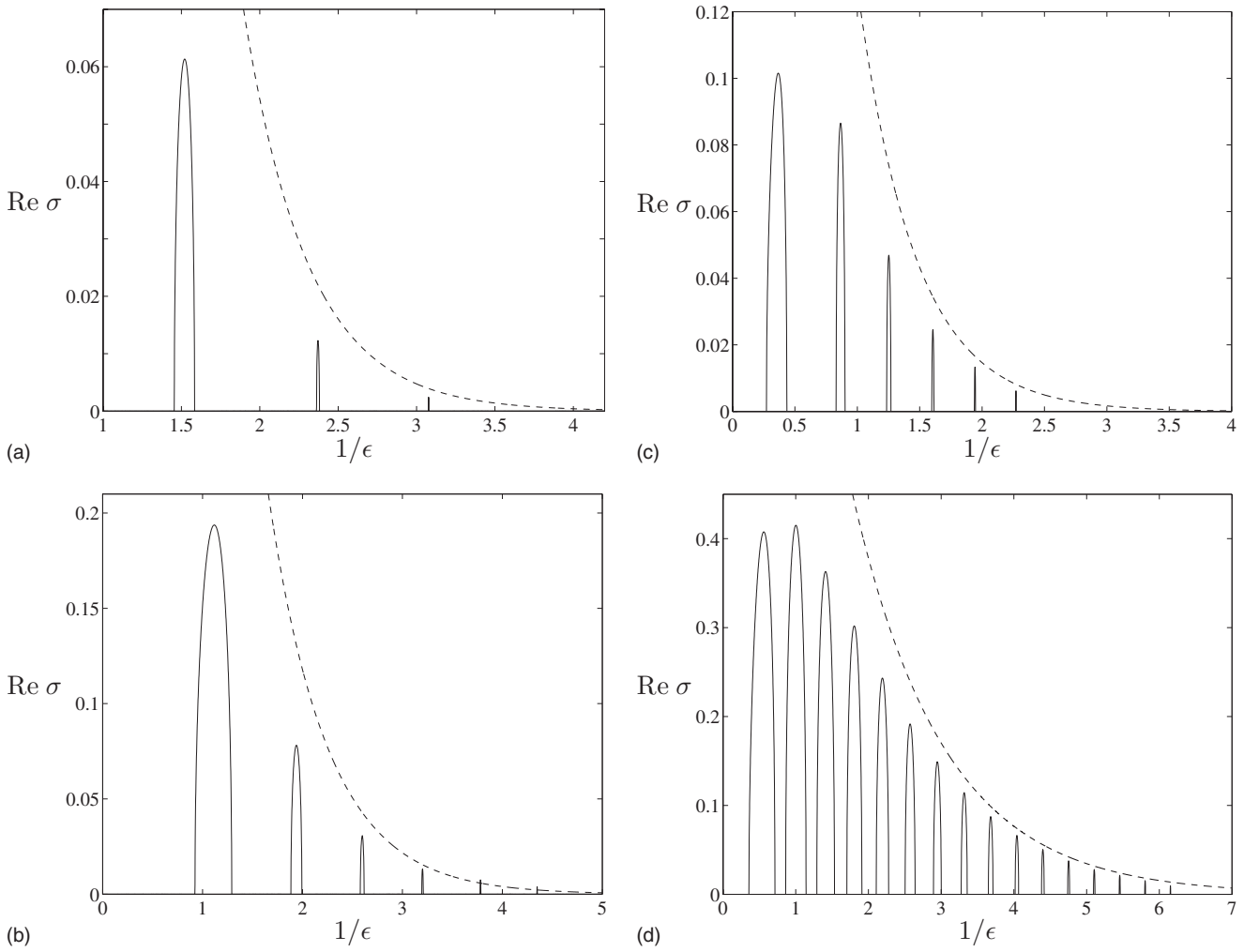

FIG. 3. Growth rates Re σ in anticyclonic flows as functions of the inverse Rossby number $1/\epsilon$ for (a) $e=1.5$, $\mu=1$; (b) $e=2$, $\mu=1$; (c) $e=2$, $\mu=0.5$; and (d) $e=4$, $\mu=1$. The growth rates computed numerically (solid lines) are compared with the asymptotic estimate of the maximum growth rates $\sigma_{\max}$ (dashed lines).

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APPENDIX A: DERIVATION OF EQUATION (2.6)

The equations that govern an inviscid stratified Boussinesq fluid rotating with constant Coriolis frequency f are

$$D_t u - fv = -p_x, \tag{A1}$$

$$D_t v + fu = -p_y, \tag{A2}$$

$$D_t w + \rho = -p_z, \tag{A3}$$

$$D_t \rho - N^2 w = 0, \tag{A4}$$

$$u_x + v_y + w_z = 0, \tag{A5}$$

where (u,v,w) are the usual Cartesian velocity components and $D_t = \partial_t + u\partial_x + v\partial_y + w\partial_z$ is the material derivative. The variable ρ is a scaled density, such that the total density is given by

$$\rho_{\text{tot}} = \rho_b + \frac{\rho_b}{g}[-N^2 z + \rho(x,y,z,t)], \tag{A6}$$

where ρ_b is the uniform background density, and N is the Brunt–Väisälä frequency. The Boussinesq approximation assumes that $|\rho_{\text{tot}} - \rho_b| \ll \rho_b$.

We seek solutions that are the sum of the elliptical flow (2.1) and the small-amplitude plane wave with time-dependent wavevector (2.3). Thus we introduce the decomposition

$$(u,v,w,\rho,p) = (ay,-bx,0,0,b(a-f)x^2/2 + a(b-f)y^2/2) + (\hat{u}(t),\hat{v}(t),\hat{w}(t),\hat{\rho}(t),\hat{p}(t))e^{i\boldsymbol{k}(t)\cdot\boldsymbol{x}} \tag{A7}$$

into Eqs. (A1)–(A5) and linearize the resulting equations. Imposing that the wavevector components satisfy Eq. (2.4) eliminates the terms that depend explicitly on x and y, leaving a system of five ordinary differential equations for $[\hat{u}(t),\hat{v}(t),\hat{w}(t),\hat{\rho}(t),\hat{p}(t)]$. This system is simplified by introducing the amplitudes of the vertical component of vorticity $\hat{\zeta} = ik\hat{v} - il\hat{u}$ and of the horizontal divergence $\hat{\delta} = ik\hat{u} + il\hat{v}$. These amplitude are found to satisfy

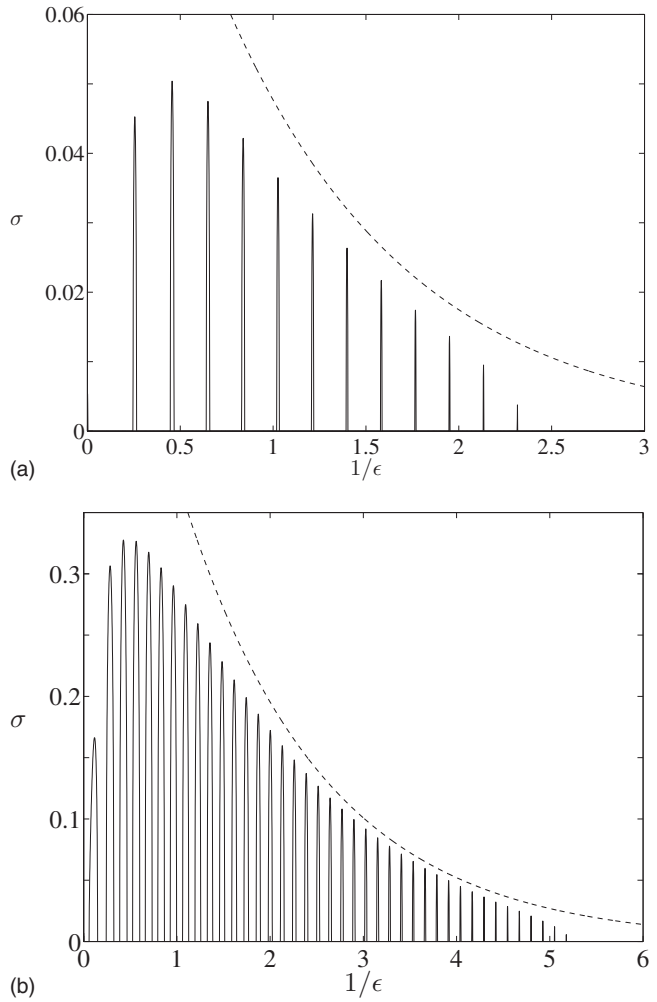

FIG. 4. Growth rates Re σ in cyclonic flows as functions of the inverse Rossby number $1/\epsilon$ for (a) $e=4$, $\mu=0.5$; and (b) $e=6$, $\mu=0.5$. The growth rates computed numerically (solid lines) are compared to the asymptotic estimate of the maximum growth rates $\sigma_{\max}$ (dashed lines).

$$\dot{\hat{\zeta}} = (a+b-f)\hat{\delta}, \tag{A8}$$

$$\dot{\hat{\delta}} = \frac{i(k^2+l^2)}{m}(\hat{\rho}+\hat{w}) + \left(f - \frac{2(k^2a+l^2b)}{k^2+l^2}\right)\hat{\zeta} + \frac{2(b-a)kl}{k^2+l^2}\hat{\delta}. \tag{A9}$$

The vertical velocity $\hat{w}$ can be eliminated from Eq. (A9) using that $\hat{\delta}+im\hat{w}=0$. To eliminate the density $\hat{\rho}$, we first obtain the conservation $\dot{\hat{q}}=0$ for the (scaled) potential vorticity $\hat{q}=N^2\hat{\zeta}+i(a+b-f)m\hat{\rho}$. Assuming that $\hat{q}(t)=\hat{q}(0)=0$ then reduces Eqs. (A8) and (A9) to two coupled equations for $\hat{\zeta}$ and $\hat{\delta}$, respectively. Eliminating $\hat{\delta}$ and removing the hat of $\hat{\zeta}$ finally leads to Eq. (2.6).

APPENDIX B: EXPONENTIAL ASYMPTOTICS

In this appendix, we compute the (exponentially small) Stokes multiplier S which quantifies the switching on of one branch of the WKB solution by the other [see Eqs. (3.6) and (3.7)] through a Stokes phenomenon.[14,18] The Stokes phenomenon is associated with the presence of complex turning points, that is, complex times where $\omega=0$. From Eq. (2.11), these are located at

$$t_n = i\,\sinh^{-1}\frac{\sqrt{1+\mu^2}}{\psi} + n\pi, \quad n=0,\pm 1,\pm 2,\ldots$$

and $\bar{t}_n$. The Stokes phenomenon occurs across Stokes lines, which are defined by $\mathrm{Re}\int_{t_n}^{t}\omega(t')dt'=0$. Our interest is in the Stokes lines crossing the interval $[-\pi/2,\pi/2]$. It is easy to see that the segment of Re $t=0$ joining t_0 to $\bar{t}_0$ is such a Stokes line; it is also the only one since for $-\pi/2<t<\pi/2$, $\mathrm{Re}\int_{t_0}^{t}\omega(t')dt' = \mathrm{Re}\int_{0}^{t}\omega(t')dt' = \int_{0}^{t}\omega(t')dt' \neq 0$ unless $t=0$.

We compute S using matched asymptotics, examining how the solution $\zeta=A(t)\exp[i\theta(t)/\epsilon]$ connects to the solution $\zeta=A(t)\{\exp[i\theta(t)/\epsilon]+S\exp[-i\theta(t)/\epsilon]\}$ as this segment is crossed.[21] To analyze the behavior of the solution in the neighborhood of t_0, we first note that

$$\omega \sim a^{1/2}e^{i\pi/4}(t-t_0)^{1/2}, \quad \text{where} \quad a=\sqrt{2(1+\mu^2)(1+\psi^2+\mu^2)}/\mu^2, \tag{B1}$$

as $t\to t_0$. We then rescale the evolution equation for ζ near t_0 by defining the inner variables

$$\tau = \epsilon^{-2/3}a^{1/3}(t-t_0) \quad \text{and} \quad Z(\tau)=\zeta(t).$$

Retaining only the leading-order terms, this transforms Eq. (2.10) into the equation

$$\frac{d^2Z}{d\tau^2} + i\tau Z = 0. \tag{B2}$$

Solutions can be written in terms of the Airy functions $\mathrm{Ai}(e^{-i\pi/6}\tau)$ and $\mathrm{Bi}(e^{-i\pi/6}\tau)$. We claim that the solution of interest is

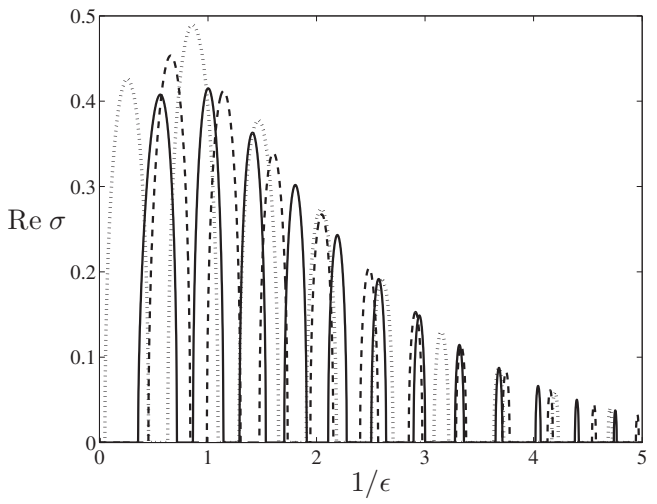

FIG. 5. Effect of the hydrostatic approximation: The growth rate Re σ is plotted as a function of the inverse Rossby number $1/\epsilon$ for an anticyclonic flow with $e=4$ and $\mu=1$, in the hydrostatic limit $N/f\to\infty$ (solid lines), for $N/f=6$ (dashed lines), and for $N/f=3$ (dotted lines).

$$Z \sim C[\mathrm{Ai}(e^{-i\pi/6}\tau) + i\ \mathrm{Bi}(e^{-i\pi/6}\tau)]. \tag{B3}$$

We verify that this solution matches $A(t)\exp[i\theta(t)/\epsilon]$ to the left of the Stokes line Re $t=0$; in doing so we find an expression for the constant C. It is convenient to verify the matching along the line ph $\tau=-5\pi/6$; this is an anti-Stokes line along which the two independent solutions of Eq. (B2) have similar orders of magnitudes. Along this line, we can use the asymptotic formulas[24]

$$\mathrm{Ai}(-z) \sim \frac{1}{\sqrt{\pi} z^{1/4}} \cos(2z^{3/2}/3 - \pi/4), \tag{B4}$$

$$\mathrm{Bi}(-z) \sim -\frac{1}{\sqrt{\pi} z^{1/4}} \sin(2z^{3/2}/3 - \pi/4), \tag{B5}$$

with $z=-\exp(-i\pi/6)\tau=\exp(5i\pi/6)\tau\to+\infty$. Thus we have

$$Z \sim \frac{Ce^{i\pi/24}}{\sqrt{\pi}\tau^{1/4}} e^{2ie^{i\pi/4}\tau^{3/2}/3}. \tag{B6}$$

On the other hand, using Eqs. (3.4) and (3.5) the solution $\zeta=A(t)\exp[i\theta(t)]$, valid in the outer region away from t_0 and to the left of the Stokes line Re $t=0$, can be written as

$$\begin{aligned}\zeta &\sim \frac{1}{\omega^{1/2}} e^{\int_{-\pi/2}^{t} p(t')dt'/2} e^{i\epsilon^{-1}\int_{-\pi/2}^{t}\{\omega(t')-\epsilon q(t')/[2\omega(t')]\}dt'} \\ &\sim \frac{e^{-i\pi/8}}{(\epsilon a)^{1/6}\tau^{1/4}} e^{\int_{\Gamma_-} p(t')dt'/2} e^{i\epsilon^{-1}\int_{\Gamma_-}\{\omega(t')-\epsilon q(t')/[2\omega(t')]\}dt'} \\ &\quad \times e^{2ie^{i\pi/4}\tau^{2/3}/3},\end{aligned} \tag{B7}$$

after using Eq. (B1). Here Γ_- denotes a contour joining $-\pi/2$ to t_0 while remaining to the left of the Stokes line Re $t=0$. Comparing Eq. (B7) with Eq. (B6) shows that ζ correctly matches Z provided that

$$C = \frac{\sqrt{\pi} e^{-i\pi/6}}{(\epsilon a)^{1/6}} e^{\int_{\Gamma_-} p(t')dt'/2} e^{i\epsilon^{-1}\int_{\Gamma_-}\{\omega(t')-\epsilon q(t')/[2\omega(t')]\}dt'}. \tag{B8}$$

We now match Z with the outer solution valid to the right of the Stokes line Re $t=0$. A connection formula for Airy functions[24] gives the alternative form

$$Z \sim 2Ce^{i\pi/3}\ \mathrm{Ai}(e^{-i5\pi/6}\tau), \tag{B9}$$

for Eq. (B3). Carrying out the matching on the anti-Stokes line ph $\tau=\exp(-i\pi/6)$, we can use the asymptotic formula for Ai in Eq. (B4) to write that

$$Z \sim \frac{Ce^{i\pi/24}}{\sqrt{\pi}\tau^{1/4}} (e^{2ie^{i\pi/4}\tau^{3/2}/3} + ie^{-2ie^{i\pi/4}\tau^{3/2}/3}) \tag{B10}$$

for $|\tau|\to\infty$. This should be matched with the form $\zeta(t)=A(t)\{\exp[i\theta(t)/\epsilon]+S\exp[-i\theta(t)/\epsilon]\}$ of the solution to the right of the Stokes line. Using Eq. (B1) gives the asymptotics

$$\begin{aligned}\zeta &\sim \frac{1}{\omega^{1/2}} e^{\int_{-\pi/2}^{t} p(t')dt'/2}\Big[e^{i\epsilon^{-1}\int_{-\pi/2}^{t}\{\omega(t')-\epsilon q(t')/[2\omega(t')]\}dt'} \\ &\quad + Se^{-i\epsilon^{-1}\int_{-\pi/2}^{t}\{\omega(t')-\epsilon q(t')/[2\omega(t')]\}dt'}\Big] \\ &\sim \frac{e^{-i\pi/8}}{(\epsilon a)^{1/6}\tau^{1/4}} e^{\int_{\Gamma_+} p(t')dt'/2} \\ &\quad \times\Big[e^{i\epsilon^{-1}\int_{\Gamma_+}\{\omega(t')-\epsilon q(t')/[2\omega(t')]\}dt'} e^{2ie^{i\pi/4}\tau^{3/2}/3} \\ &\quad + Se^{-i\epsilon^{-1}\int_{\Gamma_+}\{\omega(t')-\epsilon q(t')/[2\omega(t')]\}dt'} e^{-2ie^{i\pi/4}\tau^{3/2}/3}\Big],\end{aligned}$$

where Γ_+ denotes a contour joining $-\pi/2$ to t_0. This contour crosses the Stokes line Re $t=0$ below the singular point t_p of $p(t)$ and $q(t)$, given by $t_p=i\sinh^{-1}(1/\psi)$. Taking Eq. (B8) into account, the matching with Eq. (B10) gives the two equations

$$e^{\int_{\Gamma_-} p(t)dt/2} e^{-i\int_{\Gamma_-} q(t)/\omega(t)dt/2} = e^{\int_{\Gamma_+} p(t)dt/2} e^{-i\int_{\Gamma_+} q(t)/\omega(t)dt/2}, \tag{B11}$$

$$\begin{aligned}&ie^{\int_{\Gamma_-} p(t')dt'/2} e^{i\epsilon^{-1}\int_{\Gamma_-}\{\omega(t')-\epsilon q(t')/[2\omega(t')]\}dt'} \\ &\quad = Se^{\int_{\Gamma_+} p(t')dt'/2} e^{-i\epsilon^{-1}\int_{\Gamma_+}\{\omega(t')-\epsilon q(t')/[2\omega(t')]\}dt'}.\end{aligned} \tag{B12}$$

We can now deform the integration contours. The difference $\int_{\Gamma_+}-\int_{\Gamma_-}$ reduces to an integral along a closed contour encircling t_p. Computing the corresponding residues using Eq. (2.12) gives

$$\mathrm{Res}_{t_p} p = 1 \quad \text{and} \quad \mathrm{Res}_{t_p} \frac{q}{\omega} = -i\varsigma.$$

Taking this into account confirms that Eq. (B11) is an identity. Using $\mathrm{Res}_{t_p} p=1$ in Eq. (B12) gives the Stokes multiplier as

$$S = -ie^{2i\epsilon^{-1}\fint_{-\pi/2}^{t_0}\{\omega(t')-\epsilon q(t')/[2\omega(t')]\}dt'},$$

where the Cauchy principal value integral, denoted by $\fint$, is necessary because $q(t)$ has a pole at $t=t_p$. It follows that $|S|$, giving the instability growth rate, can be written as

$$|S| = e^{-\alpha/\epsilon+\varsigma\beta},$$

where the two constants

$$\alpha = -2i\int_0^{t_0} \omega(t)dt \quad \text{and} \quad \beta = -i\fint_0^{t_0} \frac{\varsigma q(t)}{\omega(t)}dt$$

are real, positive and independent of ς. Using Eqs. (2.11) and (2.12), they can be given the more explicit forms

$$\alpha = \frac{2}{\mu}\int_0^{\sinh^{-1}(\sqrt{1+\mu^2}/\psi)} \sqrt{1-\psi^2\sinh^2 u + \mu^2}\,du,$$

and

$$\begin{aligned}\beta = \mu\varsigma\int_0^{\sinh^{-1}(\sqrt{1+\mu^2}/\psi)} &\left(e + e^{-1} + \frac{2e}{1-\psi^2\sinh^2 u}\right) \\ &\times \frac{du}{\sqrt{1-\psi^2\sinh^2 u + \mu^2}},\end{aligned}$$

and further transformed into the convenient expressions (3.13) and (3.14).

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